Licenciatura em Gestão

Operational Research Chapter 5

2018-2019

ISEG 100 1911-2011 ANOS

100 ANOS A PENSAR NO FUTURO

5. Network Optimization

Network optimization

Terminology

- a graph or network is an ordered pair $G=(V,A)$ where $V=\{1,2,\ldots,n\}$ is a set of points called nodes or vertices and $A=\{a_1,a_2,\ldots,a_m\}$ is a set of lines connecting pairs of nodes, $a_k=(i,j)$ where $i,j \in V$.
- the lines are called arcs if they are directed (i.e. have orientation - an arrow indicates direction);
- the lines are called edges if they are undirected;
- an arc/edge $(i,j) \in A$ is incident to *i* and *j*;
- $G=(V,A)$ is a directed network if all the elements of A are arcs;
- $G=(V,A)$ is an undirected network if all the elements of A are edges;
- if A has both arcs and edges it is a mixed network (any mixed or undirected network may be converted into a directed network);

Network optimization

- if $(i,j) \in A$ is an arc, then *i* is the initial node of (i,j) and *j* is the final node of (i,j), being *j* the **sucessor** of *i* and *i* the **predecessor** of *j*.

- if $(i,j) \in A$ is an edge, then *i* and *j* are extremities of (i,j)
  and *i* and *j* are **adjacent nodes .**

- a path between two nodes is a sequence of distinct arcs/edges connecting these nodes;

- a directed path from node ***i*** to node ***j*** is a path toward node ***j***;

- an undirected path from node ***i*** to node ***j*** is a sequence of connecting arcs/edges whose direction (if any) can be either toward or away from node ***j***;

- a cycle is a path that begins and ends in the same node;

Network optimization

- two nodes are connected if the network contains at least one *undirected path* between them;
- a connected network is a network where every pair of nodes is connected;
- a tree is a connected network with no cycles;
- a spanning tree of $G=(V,A)$ is a tree with the same set of nodes V and edges belonging to A;
- to each arc/edge and each node may also be associated real parameters that are problem dependent, representing: time, capacity, distance, probabilities, supply, demand, flow, length…;
- a supply node (or source node or origin) is a node where the flow out exceeds the flow into the node;
- a demand node (or sink node or destination) a node where the flow into exceeds the flow out the node;
- a transshipment node (or intermediate node) is a node where the flow in equals the flow out, that is, there is *conservation of flow*;

Network optimization

The Minimum Cost Flow Problem (MCFP)

Let $G=(V,A)$ a directed and connected network with *at least one supply node* and *at least one demand node*, being the remaining nodes transshipment nodes. Determine how to send the available supply of a commodity through the network so as to satisfy the demand, respecting arc capacities, at minimum cost.

Data:

$G=(V,A)$ directed and connected network;

To each arc $(i,j) \in A$ two parameters are associated:

u_{ij} the arc capacity, i.e., the maximum value of flow that may traverse it (real $u_{ij} \geq 0$)
c_{ij} the cost per unit flow through the arc

To each node, $i \in V$, is associated the parameter b_i :

b_i = net flow generated at node i
= flow *out* node i – flow *into* node i

Network optimization - MCFP

The value of b_i depends on the nature of node *i*, where

$b_i>0$ if node *i* is a supply node,

$b_i<0$ if node *i* is a demand node,

$b_i=0$ if node *i* is a transshipment node.

Assumptions:

1) arc capacities are compatible with supplies and demands;

2) the problem is balanced: the total supply equals the total demand: $\sum_{i \in V} b_i = 0$

Network optimization - MCFP

Let x_{ij} be the amount of flow (units of the commodity) through arc $(i,j) \in A$ and Z the total cost of sending the available supply through the network.

The LP formulation for the MCFP is:

$$\text{Min } Z = \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$s.t. \begin{cases} \sum_{j\in V} x_{ij} - \sum_{k\in V} x_{ki} = b_i, & \forall i \in V \\ 0 \leq x_{ij} \leq u_{ij}, & \forall (i,j) \in A \end{cases}$$

For the network formulation of the MCFP:

Identify the network optimisation problem to solve; design the network $G(V,A)$.

Associate to each arc (i,j) the parameters (c_{ij},u_{ij}) and to each node i the parameter b_i identifying the supply and the demand nodes.

Define the variable x_{ij} and the objective.

Network optimization - MCFP

Properties of the MCFP

Property 1: The MCFP has, at least, a feasible solution.

Corollary: The MCFP has an optimal solution.

Property 2: A MCFP where every b_i and u_{ij} are integer values has, at least, one optimal solution with all variables assuming integer values.

Applications:

- management of passenger traffic
- production management and distribution of commodities
- personnel scheduling
- planning the distribution of fluids (water, gas, oil)

Special cases of MCFP: Transportation Problem (MCFP without transshipment nodes, and with no capacity constraints); Assignment Problem and others.

Network optimization - MCFP

Resolution methods for the MCFP

- Algorithm for LP (for example, the Simplex Method– solver/excel)

- The Network Simplex Method;

- *The Out of Kilter Method*;

- ...

Variants of the MCFP

- **Total Supply > Total Demand:**

The net flow at the sources is a maximum value that must be respected, and the constraints at the sources should be of type “≤”.

- **Total Supply < Total Demand:**

The net flow at the sinks is a minimum value that must be respected, and the constraints at the sink nodes should be of type “≥”.

Network optimization - MCFP

Resolution of the MCFP with the *Solver/Excel*

Prototype example 1 – *Distribution Unlimited Co.* (HL, & 3.4, pp. 58)

	A	B	C	D	E	F	G	H	I	J	K
1	oblem - Distribution Unlimited Co.										
2											
3											
4	arcs										
5	from	to	flow		capacity	unit cost		nodes	from-to		supply/demand
6	F1	F2	0	<=	10	200		F1	0	=	50
7	F1	DC	0			400		F2	0	=	40
8	F1	W1	0			900		DC	0	=	0
9	F2	DC	0			300		W1	0	=	-30
10	DC	W2	0	<=	80	100		W2	0	=	-60
11	W1	W2	0			300					
12	W2	W1	0			200					
13											
14		total cost	0								

=SUMPRODUCT(flow;unit_cost)

=SUMIF(from;H6;$C$6:$C$12)-SUMIF(to;H6;$C$6:$C$12)
=SUMIF(from;H7;$C$6:$C$12)-SUMIF(to;H7;$C$6:$C$12)
=SUMIF(from;H8;$C$6:$C$12)-SUMIF(to;H8;$C$6:$C$12)
=SUMIF(from;H9;$C$6:$C$12)-SUMIF(to;H9;$C$6:$C$12)
=SUMIF(from;H10;$C$6:$C$12)-SUMIF(to;H10;$C$6:$C$12)

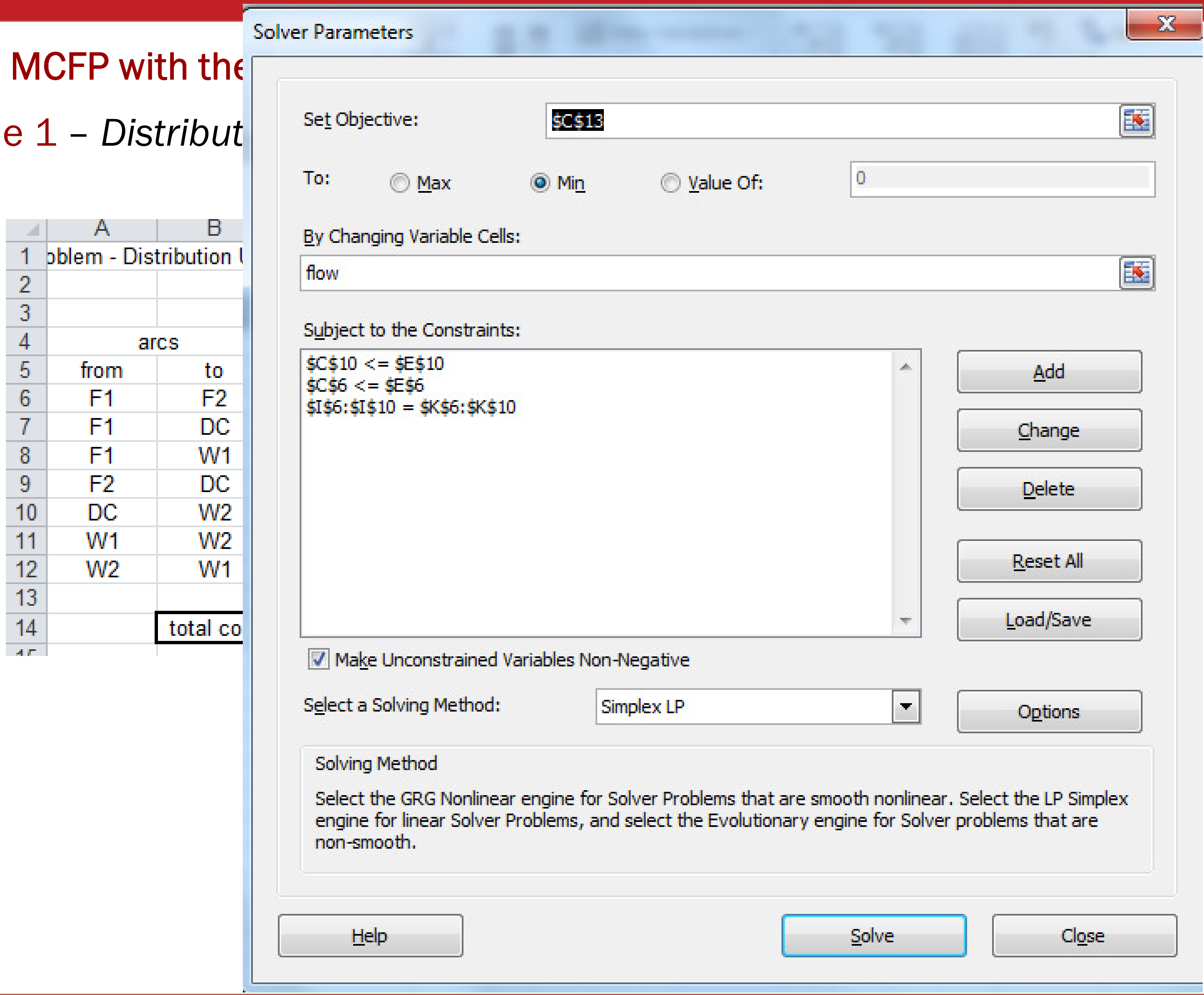
Define names: (formulas tab)

from= A6:A12
to= B6:B12
flow= C6:C12
unit_cost= F6:F12

Network optimization - MCFP

Resolution of the MCFP with the

Prototype example 1 – *Distribut*



Network optimization - SPP

The Shortest-Path Problem (SPP)

Let $G=(V,A)$ a directed and connected network with *only one origin* and *only one destination*. Associated to each arc $(i,j) \in A$ is a nonnegative real number c_{ij} that represents the length between nodes *i* and *j*. Find the shortest path (the path with the minimum total length) from the origin node to the destination node.

Data:

$G=(V,A)$ directed and connected network; $\boldsymbol{s} \in V$ the origin; $\boldsymbol{t} \in V$ the destination

$c_{ij} > 0$ length (or cost, or distance) of arc $(i,j) \in A$

For the network formulation of the SPP:

Identify the network $G=(V,A)$, as well as the origin, the destination and all the lengths

(c_{ij}) , the problem to solve and the objective.

Network optimization - SPP

For the LP formulation of the SPP:

define variables $x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is in the path} \\ 0 & \text{otherwise} \end{cases}$

Z – total length of the path from ***s*** to ***t*** (the sum of the length of all arcs in the path)

$$\text{Min Z} = \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$s.t. \begin{cases} \sum_{j:(s,j)\in A} x_{sj} = 1 \\ \sum_{i:(i,t)\in A} x_{it} = 1 \\ \sum_{j:(i,j)\in A} x_{ij} = \sum_{k:(k,i)\in A} x_{ki} \quad \forall i \in V \setminus \{s,t\} \\ x_{ij} \in \{0,1\}, \qquad \forall\, (i,j) \in A \end{cases}$$

The first two constraints ensure that the path starts at the origin, *s*, and ends at the destination, *t*. The third set of restrictions defines the remaining nodes as nodes that may be used to form the path.

Network optimization - SPP

The SPP is a particular case of the Minimum Cost Flow Problem where:

$$u_{ij} = 1, \quad \forall (i, j) \in A$$

$$b_s = 1$$

$$b_t = -1$$

$$b_i = 0, \quad \forall i \in V, \quad i \neq s, t$$

Property (integer solutions):

In the LP model for the SPP, if each $x_{ij} \in \{0,1\}$ is substituted by $x_{ij} \geq 0$, then at least one optimal solution exists with all variables assuming integer values.

Resolution of the SPP with the *Solver/Excel*

Prototype example 2 – *Seervada Park 1st* (HL, chap 9.3 pp. 364)

	A	B	C	D	E	F	G	H	I	J
1		Seervada Park								
2										
3	From	to	solution	distance		node	total		Supply/Demand	
4	O	A	0	2		O	0	=	1	
5	O	B	0	5		A	0	=	0	
6	O	C	0	4		B	0	=	0	
7	A	B	0	2		C	0	=	0	
8	A	D	0	7		D	0	=	0	
9	B	A	0	2		E	0	=	0	
10	B	C	0	1		T	0	=	-1	
11	B	D	0	4						
12	B	E	0	3						
13	C	B	0	1						
14	C	E	0	4						
15	D	E	0	1						
16	D	T	0	5						
17	E	D	0	1						
18	E	T	0	7						
19										
20		TOTAL DISTANCE		0						
21										

total
=SUMIF($A$4:$A$18;F4;$C$4:$C$18)-SUMIF($B$4:$B$18;F4;$C$4:$C$18)
=SUMIF($A$4:$A$18;F5;$C$4:$C$18)-SUMIF($B$4:$B$18;F5;$C$4:$C$18)
=SUMIF($A$4:$A$18;F6;$C$4:$C$18)-SUMIF($B$4:$B$18;F6;$C$4:$C$18)
=SUMIF($A$4:$A$18;F7;$C$4:$C$18)-SUMIF($B$4:$B$18;F7;$C$4:$C$18)
=SUMIF($A$4:$A$18;F8;$C$4:$C$18)-SUMIF($B$4:$B$18;F8;$C$4:$C$18)
=SUMIF($A$4:$A$18;F9;$C$4:$C$18)-SUMIF($B$4:$B$18;F9;$C$4:$C$18)
=SUMIF($A$4:$A$18;F10;$C$4:$C$18)-SUMIF($B$4:$B$18;F10;$C$4:$C$18)

=SUMPRODUCT(C4:C18;D4:D18)

Resolution of the SPP with the *Solver/Excel*

Prototype example 2 – *Seervada Park 1st* (HL, chap 9.3 pp. 364)

Seervada Park

From	to	solution	distance
O	A	0	2
O	B	0	5
O	C	0	4
A	B	0	2
A	D	0	7
B	A	0	2
B	C	0	1
B	D	0	4
B	E	0	3
C	B	0	1
C	E	0	4
D	E	0	1
D	T	0	5
E	D	0	1
E	T	0	7

TOTAL DISTANCE: 0

node	total
O	0
A	0
B	0
C	0
D	0
E	0
T	0

Solver Parameters

Set Objective: $D$20

To: Max / **Min** / Value Of: 0

By Changing Variable Cells: $C$4:$C$18

Subject to the Constraints:
$G$4:$G$10 = $I$4:$I$10

Add | Change | Delete | Reset All | Load/Save

☑ Make Unconstrained Variables Non-Negative

Select a Solving Method: Simplex LP | Options

Solving Method
Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.

Help | Solve | Close

Seervada Park

From	to	solution	distance
O	A	1	2
O	B	0	5
O	C	0	4
A	B	1	2
A	D	0	7
B	A	0	2
B	C	0	1
B	D	1	4
B	E	0	3
C	B	0	1
C	E	0	4
D	E	0	1
D	T	1	5
E	D	0	1
E	T	0	7

TOTAL DISTANCE: 13

node	total		Supply/Demand
O	1	=	1
A	0	=	0
B	0	=	0
C	0	=	0
D	0	=	0
E	0	=	0
T	-1	=	-1

Network optimization - MSTP

Minimum Spanning Tree Problem (MSTP)

Given an undirected and connected network, with lengths associated to the edges, choose the set of edges that represent a spanning tree (a tree including all network nodes) with minimum total length.

Data:

$G=(V,A)$ undirected network;

$c_{ij} > 0$ length (or cost, or distance) of edge $(i,j) \in A$

For the network formulation of the MSTP:

Identify the network, $G=(V,A)$, as well as all the lengths (c_{ij}) and the problem to solve.

Property : A spanning tree of a network with n nodes has the same n nodes and $n-1$ edges (no cycles).

Prim Algorithm (1957) determines the minimum spanning tree of *G*.

Objective for iteration *k* - Select the node that is not yet in the tree and is the closest to the tree. Link the node to the tree.

Repeat until all the nodes are in the tree.

Network optimization - SPP

PRIM's Algorithm

0. Input: Undirected connected network with n nodes $G=(V,A)$; Lengths of the edges;

1. Initialisation

Choose any node and the shortest edge incident on it;
Initialise the tree with the edge and respective nodes;
$k \leftarrow 2$;

2. Iteration k

If all the nodes are in the tree ($k=n$), go to step 3.
otherwise, select the shortest edge linking a node outside the tree to a node already in the tree;
Add the edge to the tree;
$k \leftarrow k+1$;
Go to step 2.

3. Draw the minimum spanning tree and determine the total length of the tree. Stop.

Network optimization - MSTP

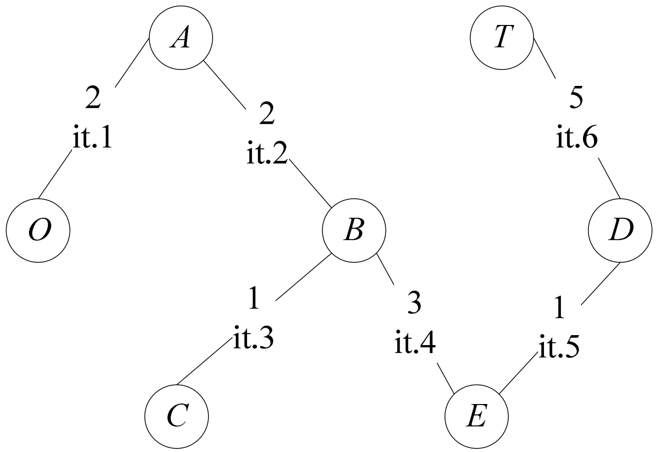
Resolution of the MSTP with the Prim Algorithm

Prototype example 2– *SEERVADA PARK* , 2nd – network with $n=7$ nodes $\Rightarrow$ n-$1=6$ iterations.

iteration	nodes in the tree	closest and adjacent node $\notin$ tree	edge length	edge to include in the tree

iteration	nodes in the tree	closest and adjacent node ∉ tree	edge length	edge to include in the tree
1	O	A	2	(O,A)
2	O	C	4	(A,B)
	A	B	**2**	
3	O	C	4	(B,C)
	A	D	7	
	B	C	**1**	
4	O	-		(B,E)
	A	D	7	
	B	E	**3**	
	C	E	4	
5	A	D	7	(E,D)
	B	D	4	
	C	-		
	E	D	**1**	
6	D	T	**5**	(D,T)
	E	T	7	

Minimum Spanning Tree



The total MST length is 14